An Introduction to Economic Modelling Techniques

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Economic Modelling and Forecasting

Kampala, February 9, 2015

DSGE models

First things first...

- D - Dynamic
- S - Stochastic
- G - General
- E - Equilibrium

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Goals for these sessions

By the end of these sessions you should:

- Understand the basic principles of DSGE modelling
- Be familiar with some of the jargon
- Know how to simulate a DSGE model using *Dynare*
- Understand *Dynare* output

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The basic approach

- Clarify costs and benefits of actions
  - Done formally in an optimisation problem
- Standard (and familiar) example: how does a household divide income between consumption and saving
- History provides examples of interesting solutions (expectations matter!)...
- History suggests that accounting for how people respond to changes can be crucial for policymakers!

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What do we care about?

- Assumption:
  households aim to attain the highest possible utility
- Different DSGE models will have different utility specifications
- Typically period utility will depend on
  - consumption C_t
  - leisure L_t or hours worked H_t
  - money M_t
- Sometimes also on
  - internal habits
  - external habits
  - other stuff...

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How do we care?

- Need to be specific about how period utility depends on relevant variables
- We have many different functional forms to choose from:
  - linear: $u(C_t) = C_t$
  - quadratic: $u(C_t) = C_t^2$
  - log: $u(C_t) = \log(C_t)$ (we'll use this today)
  - CRRA: $u(C_t) = \frac{C_t^{1-\gamma}-1}{1-\gamma}$
- We also need to be specific about how the interaction of variables affects period utility; two popular specifications are:
  - separable utility, e.g.: $u(C_t, H_t) = \frac{C_t^{1-\gamma}-1}{1-\gamma} - \frac{H_t^{1-\phi}-1}{1-\phi}$
  - non-separable utility, e.g.: $u(C_t, H_t) = \frac{(C_t^{1-\gamma}-1)}{1-\gamma} \cdot (1 - H_t)$
- Key distinction between variables and parameters

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- Key distinction between variables and parameters

What about the future?

- The reason we save (rather than consume everything today) is because we care about the future
- We shall assume that total utility depends on expected discounted values of future period utilities, i.e.

$$U = E_0 \sum_{t=0}^{+\infty} \beta^t u(C_t) = E_0 \sum_{t=0}^{+\infty} \beta^t \log(C_t)$$

- Note on jargon: β – is a parameter called the discount factor
  - What is β capturing?
- Accounting for the expectation operator E_0 in a consistent way sometimes referred to as the rational expectations revolution
  - It has profound implications (e.g. for opening anecdote...)!

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Optimisation constraints

- Absent constraints, the utility maximisation problem would have a simple solution. What would it be?
- It is typically assumed that agents can only save / invest out of income Y_t; relevant constraint is

$$C_t + I_t \leq Y_t$$

- Note on jargon: this is the budget constraint
- Since we care about GE, we still need to specify:
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Linking savings with capital

- We shall assume that I_t - i.e. all resources which are invested / saved, end up increasing the stock of capital K_t
- Capital will depreciate every period, with a fraction δ lost
- Combining these two assumptions allows us to write down the law of motion for capital

$$K_t = (1 - \delta)K_{t-1} + I_t$$

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Linking capital with production

- Capital K_t gets used in the production process with a lag
- The production function specifies the amount of output Y_t generated from the stock of capital K_{t-1}

$$Y_t = \exp(Z_t) K_{t-1}^{\alpha}$$

- Note on jargon:
  - α – is a parameter referred to as the share of capital in production (only for Cobb-Douglas production functions)
  - Z_t – is a variable called productivity
- Is there anything stochastic / uncertain in the model so far?

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Making the problem stochastic

- Our model will be stochastic, and hence a true DSGE, because we shall posit that productivity evolves according to

$$\forall t \geq 0 : Z_t = \rho Z_{t-1} + \varepsilon_t$$

where ε_t is an N.i.d.$(0, \sigma^2)$ sequence of stochastic variables

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A simplified, two-period version of the problem

- To build intuition let's assume the agent is only alive for two periods $\{0, 1\}$ and further normalise by setting $Z_0 = 0$
- Can you write down the full optimisation problem?

$$\max_{C_0, I_0, C_1, I_1} E_0 \sum_{t=0}^{1} \beta^t \log(C_t)$$

subject to

$$C_0 + I_0 = Y_0 \qquad Y_0 = \exp(Z_0) K_{-1}^{\alpha}$$

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- Why did we write the resource constraints with equality?
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A simplified, two-period version of the problem (ctd.)

- What we end up with is

$$\max_{C_0, I_0} \log(C_0) + \beta E_0 \log\big(\underbrace{\exp(\varepsilon_1)\left((1-\delta)K_{-1} + I_0\right)^{\alpha}}_{C_1}\big)$$

subject to

$$C_0 + I_0 = K_{-1}^{\alpha}$$

- We can further simplify this using

$$E_0 \log(C_1) = E_0 \left[\log(\exp(\varepsilon_1)) + \alpha \log((1-\delta)K_{-1} + I_0)\right]$$

where we have exploited the properties of the log function:

$$\log(XY) = \log(X) + \log(Y) \qquad \log X^{\alpha} = \alpha \log A$$

- Because ε_1 is a mean 0 random variable we can simply evaluate the expectation operator and write

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- Slope of the budget constraint: -1
- Iso-utility curves are formed of pairs (C_0, I_0) satisfying

$$\log(C_0) + \alpha\beta \log\left((1-\delta)K_{-1} + I_0\right) = U$$

- Solving out for I_0 we obtain

$$I_0(C_0; U) = \left(\exp\left(\frac{U - \log(C_0)}{\alpha\beta}\right) - (1-\delta)K_{-1}\right)$$

- The slope of the iso-utility curve will be equal to

$$\begin{aligned}\frac{\partial I_0(C_0;U)}{\partial C_0} &= -\exp\left(\frac{U-\log(C_0)}{\alpha\beta}\right)\frac{1}{\alpha\beta C_0} \\ &= -\exp\left(\log\left((1-\delta)K_{-1}+I_0\right)\right)\frac{1}{\alpha\beta C_0} = -\frac{K_0}{\alpha\beta C_0}\end{aligned}$$

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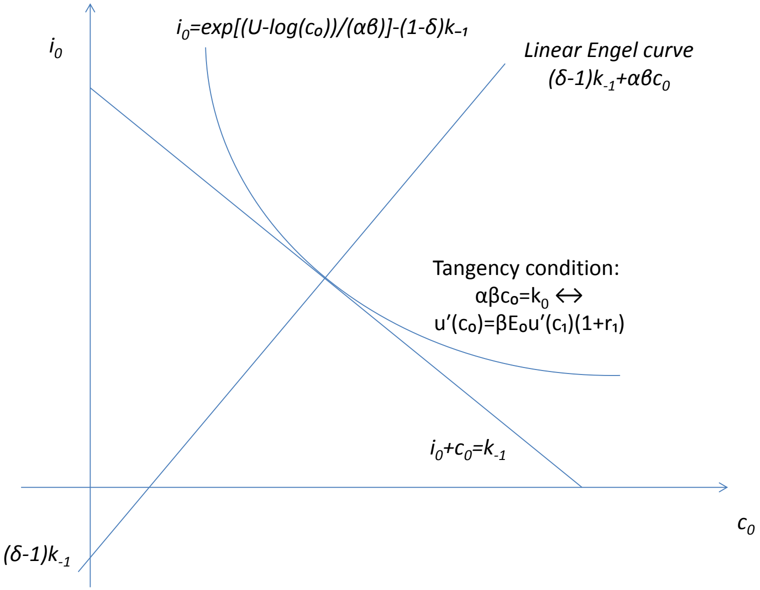
Graphical analysis of the two-period problem



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The consumption Euler equation

- In equilibrium: the iso-utility curve and the budget constraint have to have identical slope; this tangency condition is

$$-1 = -\frac{K_0}{\alpha\beta C_0}$$

- Exploiting the fact that $C_1 = Y_1$ we can rewrite this as

$$\frac{1}{C_0} = \beta E_0 \frac{1}{C_1}\frac{\alpha Y_1}{K_0}$$

or more generally as the *consumption Euler equation*

$$u'(C_0) = \beta E_0 u'(C_1)(1 + r_1)$$

This relationship is also referred to as the *dynamic IS curve*

- We used the fact that the rate of return on investment is

$$\begin{aligned}(1 + r_1) &= (\partial Y_1 / \partial I_0) = \partial \exp(Z_1)\left((1-\delta)K_{-1} + I_0\right)^{\alpha} / \partial I_0 \\ &= \exp(Z_1)\alpha\left((1-\delta)K_{-1} + I_0\right)^{\alpha-1} = \exp(Z_1)\alpha K_0^{\alpha-1} = \alpha Y_1 / K_0\end{aligned}$$

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$$\begin{aligned}(1 + r_1) &= (\partial Y_1 / \partial I_0) = \partial \exp(Z_1) \left((1 - \delta) K_{-1} + I_0\right)^{\alpha} / \partial I_0 \\ &= \exp(Z_1) \alpha \left((1 - \delta) K_{-1} + I_0\right)^{\alpha - 1} = \exp(Z_1) \alpha K_0^{\alpha - 1} = \alpha Y_1 / K_0\end{aligned}$$

The consumption Euler equation: intuition

- The cost of a marginal increase in investment: $u'(C_0)$
- The expected benefit: $\beta E_0 u'(C_1)(1 + r_1)$
- The derivations show that the expected real interest rate depends on the properties of the productivity process

$$E_0(1 + r_1) = E_0 \exp(Z_1) \alpha K_0^{\alpha - 1}$$

- Combined with the consumption Euler equation

$$u'(C_0) = \beta E_0 u'(C_1)(1 + r_1)$$

  this then implies that:
  - Higher expected productivity ($E_0 \exp(Z_1) \uparrow$) translates into higher expected rates
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The Euler equation: link to New-Keynesian models

- In models with inflation, the Fisher parity (an identity linking real and nominal interest rates and inflation)

$$1 + r_1 \equiv \frac{1 + i_1}{1 + \pi_1}$$

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- By the *exact same mechanism* as in our two-period model, higher expected inflation results in higher consumption today!
- Importantly, higher nominal interest rates i_1 would typically lead to lower consumption consumption today, in line with the standard interest rate channel of monetary policy transmission

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The resource allocation problem: general case

- We now consider the case in which the consumer / investor doesn't know his time of death with certainty
- We can formally write the problem as

$$\max_{\{C_t, I_t\}} U = \max_{\{C_t, I_t\}} E_0 \sum_{t=0}^{+\infty} \beta^t \log(C_t)$$

subject to

$$\begin{aligned} C_t + I_t &= Y_t \\ Y_t &= \exp(Z_t) K_{t-1}^{\alpha} \\ K_t &= (1-\delta) K_{t-1} + I_t \\ Z_t &= \rho Z_{t-1} + \varepsilon_t \end{aligned}$$

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- We proceed by making the problem simpler
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- The 1-1 mapping between I_t and K_t implied by the capital accumulation equation $I_t = K_t - (1-\delta)K_{t-1}$, means we can equivalently maximise over K_t rather than I_t
- For a while, we can also ignore the equation $Z_t = \rho Z_{t-1} + \varepsilon_t$ as Z_t is unaffected by C_t and K_t (ε_t is exogenous)
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Lagrange multipliers: the finite case

- Setup: maximise a function $U(X,Y)$ with respect to X and Y, subject to the constraint $PX + QY = B$
- The Lagrange multiplier approach to finding a solution
  1. Define the Lagrangian $\mathcal{L}(X,Y,\lambda)$ as
  $$\mathcal{L}(X,Y,\lambda) = U(X,Y) - \lambda(PX + QY - B)$$
  where λ is called a Lagrange multiplier
  2. Differentiate $\mathcal{L}(X,Y,\lambda)$ w.r.t. X, Y and λ and equate to 0
  $$\frac{\partial \mathcal{L}}{\partial X} = \frac{\partial U}{\partial X} - \lambda P = 0 \quad \Leftrightarrow \quad \mathcal{L}_x = U_x - \lambda P = 0$$
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  These equations are called the first-order conditions (FOCs)
  3. Use the equations to solve for X and Y. For us, they imply
  $$\frac{U_x}{U_y} = \frac{P}{Q} \Leftrightarrow U_xQ = U_yP$$

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Lagrange multipliers: a simple example

- To ensure that we understand how the technique of Lagrange multipliers works, let's apply it to a specific example
- Exercise 1: Find the maximum of $U(X, Y) = XY + 2X$ subject to the constraint $4X + 2Y = 60$

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- Solution (make sure you can replicate it): $\{X, Y\} = \{8, 14\}$

Lagrange multipliers: the finite case (ctd.)

- What if we have more constraints?
- Simply apply the procedure before, but introduce a separate Lagrange multiplier ($\lambda_1, \lambda_2, \ldots$) for each constraint

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  1. Since $\beta > 0$ we can divide all the equations through by β^t
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- We can now eliminate the Lagrange multiplier λ_t from the first-order conditions (FOCs)
- Plugging the first FOC into the second, we immediately arrive at the familiar consumption Euler equation

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \left[\alpha \exp(Z_{t+1}) K_t^{\alpha-1} + (1-\delta) \right]$$

or more generally

$$u'(C_t) = \beta E_t u'(C_{t+1}) \left[R_{t+1} + (1-\delta) \right]$$

where we have exploited the fact that the gross rate of interest $R_{t+1} \equiv 1 + r_{t+1} = \alpha Y_{t+1} / K_t = \alpha \exp(Z_{t+1}) K_t^{\alpha-1}$ as shown in the two period model

- Why does the dynamic IS curve now additionally feature a $(1-\delta)$ term that was absent in the 2 period setup?

Applying Lagrange multipliers to the general problem (ctd.)

- We can now eliminate the Lagrange multiplier λ_t from the first-order conditions (FOCs)
- Plugging the first FOC into the second, we immediately arrive at the familiar consumption Euler equation

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \left[\alpha \exp(Z_{t+1}) K_t^{\alpha-1} + (1-\delta)\right]$$

  or more generally

$$u'(C_t) = \beta E_t u'(C_{t+1}) \left[R_{t+1} + (1-\delta)\right]$$

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$$Z_t = \rho Z_{t-1} + \varepsilon_t .$$

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