SOLUTIONS TO EXERCISES ON:
AN INTRODUCTION TO DSGE MODELS

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Exercise 1. Straight from the formula in the presentation, Arrow security prices are given by

$$Q_n = \beta \pi_n \frac{\mathbf{y}_0}{\mathbf{y}_n}.$$

Since we know β and π_n all that we need to compute are the aggregate endowments in states 0 (period 0) and states 1 and 2 (period 1). Summing over y^i in the table, we obtain

$$\begin{aligned}
\mathbf{y}_0 &= 2+6+1 = 9 \\
\mathbf{y}_1 &= 5+6+4 = 15 \\
\mathbf{y}_2 &= 2+3+6 = 11.
\end{aligned}$$

Accordingly, the Arrow security prices are given by

$$\begin{aligned}
Q_1 &= \beta \pi_1 \frac{\mathbf{y}_0}{\mathbf{y}_1} = 0.99 \cdot 0.6 \cdot \frac{9}{15} = 0.3564 \\
Q_2 &= \beta \pi_2 \frac{\mathbf{y}_0}{\mathbf{y}_2} = 0.99 \cdot 0.4 \cdot \frac{9}{11} = 0.324.
\end{aligned}$$

Since Arrow security 1 pays a unit of the consumption good in state 1, and Arrow security 2 pays a unit of the consumption good in state 2, therefore a portfolio consisting of both Arrow securities will pay a unit of the consumption good with certainty, i.e. in both states in period 1. By no arbitrage, the price of the riskless bond therefore has to equal

$$Q = Q_1 + Q_2 = 0.3564 + 0.324 = 0.6804.$$

Since the riskless bond pays a unit of the consumption good, therefore its *gross* rate of return is given by

$$R = \frac{1}{Q} \approx 1.4697$$

i.e. the net riskless bond offers a real rate of return equal to 46.97%.

To compute the consumption of all agents we can simply use the formulae in the slides:

$$c_0^i = \frac{1}{1+\beta}\left(y_0^i + \sum_{m\in\{1,2\}} Q_m y_m^i\right) \quad \text{and} \quad c_n^i = \frac{\beta\pi_n/Q_n}{1+\beta}\left(y_0^i + \sum_{m\in\{1,2\}} Q_m y_m^i\right).$$

Since we've also shown that

$$Q_n = \beta\pi_n \frac{\mathbf{y}_0}{\mathbf{y}_n} \Rightarrow \beta\pi_n/Q_n = \frac{\mathbf{y}_n}{\mathbf{y}_0}$$

therefore consumption will be given by

$$c_0^i = \frac{1}{1+\beta}\left(y_0^i + \sum_{m\in\{1,2\}} Q_m y_m^i\right) \quad \text{and} \quad c_n^i = \frac{\mathbf{y}_n}{\mathbf{y}_0}\frac{1}{1+\beta}\left(y_0^i + \sum_{m\in\{1,2\}} Q_m y_m^i\right).$$

Of course, the term in the brackets is actually equal to each agent's wealth. Plugging in the numbers we find that the respective wealth levels equal:

$$\begin{aligned}
\text{For agent 1} &: \quad 2 + 0.3564\cdot 5 + 0.324\cdot 2 = 4.43\\
\text{For agent 2} &: \quad 6 + 0.3564\cdot 6 + 0.324\cdot 3 = 9.1104\\
\text{For agent 3} &: \quad 1 + 0.3564\cdot 4 + 0.324\cdot 6 = 4.396.
\end{aligned}$$

This immediately establishes that agent 2 is the richest one in our model.

Furthermore, the weights in front of the brackets are equal to

$$\frac{1}{1+\beta} = 0.502513 \qquad \frac{\mathbf{y}_1}{\mathbf{y}_0}\frac{1}{1+\beta} = 0.837521 \qquad \frac{\mathbf{y}_2}{\mathbf{y}_0}\frac{1}{1+\beta} = 0.614182.$$

Doing the multiplication, we can then immediately find that the consumptions (rounded to 2 decimal places) are

i	c_0^i	c_1^i	c_2^i
1	2.23	3.71	2.72
2	4.58	7.63	5.60
3	2.20	3.66	2.68

The sum of the (unrounded) columns equals the total consumption in period 0 and in states 1 and 2 in period 1 respectively. Summing the numbers we find that these are: $9, 15$ and 11. Since these are exactly equal to the aggregate endowments, therefore we see that consumption markets, and by extension also asset markets, have to clear.

To figure out agent 1's pattern of asset trade, we can simply compare his endowments with his consumption in each of the states. In period 0 agent 1's consumption is 2.23 and exceeds his endowment of 2. The same is true in state 2, where the endowment is equal to 2 and the consumption is equal to 2.72. In contrast, the consumption in state 1, which equals 3.71 is smaller than the endowment of 5. Putting all of these together, we see that Agent 1 sells Arrow securities paying a unit of consumption in state 1 and uses the proceeds to buy consumption in the current period along with the Arrow security paying off in state 2.

The ratio of consumption in state 1 to consumption in state 2 for all agents is 1.363636. It exactly equals the ratio of aggregate endowments $15/11 = 1.3636$. We see that agents have perfectly cross-insured, i.e. their consumptions move exactly in parallel!

Exercise 2. The procedure suggested we should get rid of the time subscripts on all variables, set the shock ε_t^Y to 0 and omit all the expectations operators; doing so gives us

$$\frac{1}{C} = \beta \frac{1}{C}[1 + R]$$

$$C = Y$$

$$\log(Y) = \rho \log(Y)$$

This is a system of 3 equations in C, Y, R; we need to solve it - i.e. express their values in terms of parameters β and ρ. To solve the system, we start with the law of motion for output

$$\log(Y) = \rho \log(Y)$$

If $\rho \neq 1$, then this implies that $\log(Y) = 0$ and so $Y = 1$.

The system then simplifies to

$$\frac{1}{C} = \beta \frac{1}{C}[1 + R]$$

$$C = 1$$

The first equation can then be immediately solved for R

$$R = \frac{1}{\beta} - 1$$

We have thus found the deterministic SS of our model: $C = Y = 1$ and $R = \frac{1}{\beta} - 1$. Plugging in $\beta = 0.99$ we find that $R \approx 0.01$. Since the model is quarterly this implies an interest rate of 4%, which is broadly in line with the data.

Exercise 3. We start with some definitions:

- A linear equation is one in which any endogenous variable X (i.e. C,Y or R in the RBC model) or shock (i.e. ε) only appears in expressions such as $\pm\gamma X$, where γ stands for a model parameter (i.e. β or ρ) or constant
- A linear model is one in which all model equations are linear

Non-linear equations can potentially obscure the underlying economic intuition and log-linearisation is a useful technique for deriving *simpler*, but *approximate*, versions of the model equations (i.e. we're sacrificing accuracy for clarity). In the past models *had to* be log-linearised by hand before being solved; while this is no longer necessary, log-linearisation is still widely used.

Log-linearisation consists of two steps:

(1) Changing variables to logs of the existing ones

(2) Linearisation of the resulting model

To show how to change variables, we use the Euler equation (the other two are trivial)

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \left[1 + R_t\right].$$

Since $\log(\cdot)$ is the inverse function of $\exp(\cdot)$, therefore

$$X = \exp(\log(X))$$

and so we can rewrite the consumption Euler equation as

$$\frac{1}{\exp(\log(C_t))} = \beta E_t \frac{1}{\exp(\log(C_{t+1}))} \cdot \left[\exp(\log(1 + R_t))\right]$$

where we have chosen to log-linearise the gross interest rate $1 + R_t$, rather than the net one (equal to the gross minus 1). We can then define a new set of variables

$$c = \log(C) \qquad\qquad r = \log(1 + R)$$

and rewrite the equation above as

$$\frac{1}{\exp(c_t)} = \beta E_t \frac{\exp(r_t)}{\exp(c_{t+1})}.$$

It is immediately clear that the transformed Euler equation is not linear in the new variables. We thus replace it using a linear, but approximate, version! To do so, we shall use a first order Taylor series approximation

$$f(x) \approx f(\bar{x}) + f'(\bar{x})(x - \bar{x})$$

In our applications:

- $f(\cdot)$ will represent an equilibrium or market clearing condition (in logs), here

$$f\left(c_t, c_{t+1}, r_t\right) = \frac{1}{\exp\left(c_t\right)} - \frac{\beta \exp\left(r_t\right)}{\exp\left(c_{t+1}\right)}$$

- x will consist of all variables in that equation (e.g. in the example above $x = (c_t, c_{t+1}, r_t)$)
- $f(\bar{x})$ and $f'(\bar{x})$ will denote the equilibrium condition f and its gradient f' evaluated in $\bar{x}$, which we set to the steady state

Note that we can always assume that $f(x_t) = 0$ as we can always put all the expressions on one side of any equation! To linearise, we now need to find the coefficients

$$f_{c_t}(c, c, r) \qquad f_{c_{t+1}}(c, c, r) \qquad f_{r_t}(c, c, r)$$

As a first step we can compute

$$f_{c_t}\left(c_t, c_{t+1}, r_t\right) = -\frac{1}{\exp\left(c_t\right)} \quad f_{c_{t+1}}\left(c_t, c_{t+1}, r_t\right) = \frac{\beta \exp\left(r_t\right)}{\exp\left(c_{t+1}\right)} \quad f_{r_t}\left(c_t, c_{t+1}, r_t\right) = -\frac{\beta \exp\left(r_t\right)}{\exp\left(c_{t+1}\right)}$$

We can then exploit the fact that in the steady state

$$1 = \beta \exp\left(r\right)$$

to simplify the preceding expressions as

$$f_{c_t}(c, c, r) = -\frac{1}{\exp\left(c\right)} \qquad f_{c_{t+1}}(c, c, r) = \frac{1}{\exp\left(c\right)} \qquad f_{r_t}(c, c, r) = -\frac{1}{\exp\left(c\right)}$$

Combining all the information we then get

$$f\left(c_t, c_{t+1}, r_t\right) \approx f\left(c, c, r\right) - \frac{1}{\exp\left(c\right)}\left(c_t - c\right) + \frac{1}{\exp\left(c\right)}\left(c_{t+1} - c\right) - \frac{1}{\exp\left(c\right)}\left(r_t - r\right)$$

Exploiting $f(c, c, r) = 0$, multiplying by $-\exp(c)$, and since we have shown above that $r = -\log(\beta)$ we end up with (note the reappearance of the expectation operator E_t)

$$(c_t - c) - E_t (c_{t+1} - c) + (r_t - r) \approx 0$$

or more compactly

$$E_t \Delta c_{t+1} \approx \hat{r}_t$$

where we've defined

$$\hat{r}_t \equiv r_t - r = r_t + \log(\beta).$$

As for the logs, we take them before linearising to ensure that the impulse responses end up being in percentage deviations of steady state values. Specifically,

$$\log(X_t) - \log(X) = \log\left(1 + \frac{X_t - X}{X}\right) \approx \frac{X_t - X}{X}.$$