BANK OF ENGLAND

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An Introduction to DSGE Models

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Title

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DSGE models

- First things first...
  - D - Dynamic
  - S - Stochastic
  - G - General
  - E - Equilibrium

Goals for these sessions

- By the end of these two sessions you should:
    - Be able to solve simple heterogenous agent DSGE models
    - Have an understanding of some of the key underlying concepts
        - Complete vs incomplete markets
        - Heterogeneous vs representative agent models
        - Links between utility specifications and choice axioms
        - Consumption-Euler equation
- Tomorrow: basic properties of the NK model

The basic approach

- Clarify costs and benefits of actions
  - Done formally in an optimisation problem
- Standard (and familiar) example: how does a household divide income between consumption and saving
- History provides examples of interesting solutions (expectations matter!)...
- History suggests that accounting for how people respond to changes can be crucial for policymakers!

A static deterministic general equilibrium model

- Initially, we shall keep things simple and solve a model which is
  - Static: i.e. there will be only one time-period, $t \equiv 1$
  - Deterministic: i.e. everything will be known at the time of making the decision
  - General Equilibrium: i.e. no agent will be able to improve their situation by unilaterally changing their behaviour
- To make things a little bit harder, we will consider a multiple good, heterogeneous agent model
  - I.e. there will be many goods traded and we will allow for differences between consumers
- Assumption:
  - Every household aims to attain the highest possible utility
  - Jargon: agent = consumer = household

Utility

- We will denote consumer i's consumption of good n by c_n^i where $i \in I$ and $n \in \{0, \ldots, N\}$
- Need to be specific about agent i's utility function
- We have many different functional forms to choose from
  - linear: $u(c_0^i, c_1^i, \ldots, c_N^i) = \gamma_0^i c_0^i + \gamma_1^i c_1^i + \ldots + \gamma_N^i c_N^i$
  - quadratic: $u(c_0^i, c_1^i, \ldots, c_N^i) = \gamma_0^i \left(c_0^i\right)^2 + \ldots + \gamma_N^i \left(c_N^i\right)^2$
  - log: $u(c_0^i, c_1^i, \ldots, c_N^i) = \gamma_0^i \log\left(c_0^i\right) + \ldots + \gamma_N^i \log\left(c_N^i\right)^2$
  - CRRA: $u(c_0^i, c_1^i, \ldots, c_N^i) = \sum_{n \in \{0,\ldots,N\}} \frac{\left(c_n^i\right)^{1-\gamma_n^i} - 1}{1-\gamma_n^i}$
- More broadly, we can have
  - separable utility: $u(c_0^i, c_1^i, \ldots, c_N^i) = f_0\left(c_0^i\right) + f_1\left(c_1^i\right) + \ldots + f_N\left(c_N^i\right)$
  - non-separable utility: any utility function which is not separable
    - E.g. $u(c_0^i, c_1^i) = c_0^i \cdot c_1^i$
- Key distinction between variables and parameters

Notes on utility

- The setup so far may seem terribly ad hoc:
  - No independent evidence that utility exists
  - No way of measuring utility
  - Different choices of utility functions could potentially lead to very different conclusions
- These objections were forcefully raised by Walras (1834-1910) and Pareto (1848-1923)

Notes on utility (ctd)

- Samuelson's (1938) "Note on the pure theory of consumer's behaviour" provided some respite
- Samuelson was
  - suspicious of the ad hoc and unobserved notion of utility
  - interested in the simplest model of choice capable of making positive predictions about consumer decisions
- The answer he provided (sharpened by Houthakker (1950)) became known as GARP (Generalised Axiom of Revealed Preference)
- A consumer is said to satisfy GARP if having chosen **B** when **C** was available, and having chosen **A** when **B** was available, she cannot strictly prefer **C** to **A**

Notes on utility (ctd)

- Afriat (1967) proved a remarkable result linking GARP to expected utility:
  - Any GARP consumer behaves exactly as if she had a continuous, concave and strongly monotone utility function underlying her decisions
- Von Neumann and Morgenstern (1944) focussed on probabilistic lotteries and showed that under the continuity and independence axioms
  - A GARP consumer behaves as if she was evaluating lotteries based on expected utilities

Notes on utility - A summary

- Positive spin: the expected utility formulation, with a continuous, concave and strongly monotone period utility function may not be as ad hoc as it initially seemed
- Negative spin: since utility is unobservable, we should be cautious about implications which don't follow from continuity, concavity or strong monotonicity
  - Behavioural evidence on continuity and independence axioms (crucial in the dynamic context) is at best mixed!

The optimisation problem

- Consumer $i \in I$ decides on consumption of $N+1$ goods to maximise utility

$$\max_{c_0^i, c_1^i, \ldots, c_N^i} \left\{ \gamma_0 u\left(c_0^i\right) + \gamma_1 u\left(c_1^i\right) + \ldots + \gamma_N u\left(c_N^i\right) \right\}$$

$$s.t. \quad : \quad \sum_{n \in \{0,1,\ldots,N\}} p_n c_n^i = \sum_{n \in \{0,1,\ldots,N\}} p_n y_n^i$$

  - c_n^i denotes agents i's consumption of good n
  - y_n^i denotes agent i's endowment of good n
  - p_n denotes the price of good n

- Key questions:
  - What does the consumer know? What does he need to solve for?
  - Are the consumers different? In what way?

- Assumptions
  - There is a market for each good n (markets are complete)
  - To fix attention / simplify, we shall set $u(\cdot) = \log(\cdot)$

Solving the heterogenous agent model

- The assumption of log-utility implies that the problem solved by consumer i is

$$\max_{c_0^i, c_1^i, \ldots, c_N^i} \left\{ \gamma_0 \log\left(c_0^i\right) + \gamma_1 \log\left(c_1^i\right) + \ldots + \gamma_N \log\left(c_N^i\right) \right\}$$

$$s.t. \quad : \sum_{n \in \{0,1,\ldots,N\}} p_n c_n^i = \sum_{n \in \{0,1,\ldots,N\}} p_n y_n^i$$

- To solve the model we shall:
  1. Characterise how much of good n agent i would like to consume conditional on prices $p_1, p_2, \ldots, p_n$
     - These solutions will define the excess demand / supply schedules
  2. Find prices $p_1, p_2, \ldots, p_n$ such that the resulting quantity demanded by all consumers equals the quantity supplied (this is the GE part)
  3. Plugging $p_1, p_2, \ldots, p_n$ back into the formulae derived in 1. will give us the actual amounts of each good consumed in equilibrium

Individual excess demand / supply schedules

- To solve the model we first characterise the consumption level which each agent would choose conditional on prices $p_1, p_2, \ldots, p_n$
  - How can we do that?
- There are several techniques for dealing with maximisation problems of this type; we will use Lagrange multipliers

Lagrange multipliers: the finite case

- Setup: maximise a function $U(X, Y)$ with respect to X and Y, subject to the constraint $PX + QY = B$
- The Lagrange multiplier approach to finding a solution

1. Define the Lagrangian $\mathcal{L}(X, Y, \lambda)$ as

$$\mathcal{L}(X, Y, \lambda) \equiv U(X, Y) - \lambda(PX + QY - B)$$

   where λ is called a Lagrange multiplier

2. Differentiate $\mathcal{L}(X, Y, \lambda)$ w.r.t. X, Y and λ and equate to 0

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial X} &= \frac{\partial U}{\partial X} - \lambda P = 0 & \Leftrightarrow \quad \mathcal{L}_x &= U_x - \lambda P = 0 \\
\frac{\partial \mathcal{L}}{\partial Y} &= \frac{\partial U}{\partial Y} - \lambda Q = 0 & \Leftrightarrow \quad \mathcal{L}_y &= U_y - \lambda Q = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= PX + QY - B = 0 & \Leftrightarrow \quad \mathcal{L}_\lambda &= PX + QY - B = 0
\end{aligned}$$

   These equations are called the first-order conditions (FOCs)

3. Use the equations to solve for X and Y. For us, they imply

$$\frac{U_x}{U_y} = \frac{P}{Q} \Leftrightarrow U_x Q - U_y P = 0$$

Lagrange multipliers: a simple example

- To ensure that we understand how the technique of Lagrange multipliers works, let's apply it to a specific example:
  - Find the maximum of $U(X, Y) = XY + 2X$ subject to the constraint $4X + 2Y = 60$
- Solution: $\{X, Y\} = \{8, 14\}$

Solving agent i's optimisation problem

- We can now apply Lagrange multipliers to the optimisation problem solved by consumer i

$$\max_{c_0^i, c_1^i, \ldots, c_N^i} \left\{ \gamma_0 \log\left(c_0^i\right) + \gamma_1 \log\left(c_1^i\right) + \ldots + \gamma_N \log\left(c_N^i\right) \right\}$$

$$s.t. \quad : \sum_{n \in \{0,1,\ldots,N\}} p_n c_n^i = \sum_{n \in \{0,1,\ldots,N\}} p_n y_n^i$$

- What is consumer i's optimum expenditure on the consumption of good n?

Individual excess demand / supply schedules - solution

- The desired expenditure on good n by consumer i is given by

$$\forall n \in \{0, \ldots, N\} : p_n c_n^i = \frac{\gamma_n}{\sum_{m \in \{0,1,\ldots,N\}} \gamma_m} \left(\sum_{m \in \{0,1,\ldots,N\}} p_m y_m^i \right)$$

  - What determines whether agent i buys/sells good n in the market?

- How does the quantity consumed depend on the price of good n?
  - What is the intuition behind the formula above?

Market clearing

- We have markets for $N + 1$ different goods types $n \in \{0, \ldots, N\}$
- We have I agents, each of whom would like to consume c_n^i
- To ensure markets are in equilibrium, what do we need to impose?
- The corresponding market clearing conditions are

$$\forall n \in \{0, 1, \ldots, N\} : \sum_{i \in I} c_n^i = \sum_{i \in I} y_n^i = \mathbf{y}_n$$

  - How can we use this condition to solve for equilibrium goods prices $p_1, p_2, \ldots, p_n$?

Equilibrium prices

- Solution: letting $Q_n \equiv p_n / p_0$ and defining the aggregate endowment of good n as $\mathbf{y}_n \equiv \sum_{i \in I} y_n^i$ we can show

$$\forall n > 0 : Q_n = \frac{\gamma_n \mathbf{y}_0}{\gamma_0 \mathbf{y}_n}$$

  - The price we're dividing by (i.e. p_0) is called the numeraire
  - Why do we need to divide by p_0 instead of simply solving for it?
- Relative prices are pinned down by a combination of aggregate endowments $\mathbf{y}$ and the (common) preference parameters γ_n
  - What is the economic intuition?

Link to dynamic stochastic (general equilibrium) models

- Why should we care about static deterministic models?
  - Famous insight of Arrow (1964) and Debreu (1959): uncertainty and time can easily be incorporated in the previous framework!
- Specifically, we can re-interpret our model as one with
  - Two periods (for simplicity; no actual constraint on the number)
  - N possible future outcomes / states next period $\{1, 2, \ldots, N\}$
  - One type of consumption good (again, only for simplicity)

$$c_0^i = \text{consumption of agent } i \text{ in the initial period}$$

$$c_n^i = \text{consumption of agent } i \text{ in state } n \in \{1, \ldots, N\} \text{ in period 2}$$

- We can also set the utility weights γ_n equal to (why?)

$$\gamma_0 = 1 \qquad \gamma_n = \beta\pi_n$$

  - We shall call β the discount factor, and π_n will denote the probability of state n occurring

A DSGE model

- Agent i's optimisation problem then becomes

$$\max_{c_0^i, c_1^i, \ldots, c_n^i} \left\{ \log\left(c_0^i\right) + \beta \sum_{n \in \{1,\ldots,N\}} \pi_n \log\left(c_n^i\right) \right\}$$

$$s.t. \quad : \quad c_0^i + \sum_{n \in \{1,\ldots,N\}} Q_n c_n^i = y_0^i + \sum_{n \in \{1,\ldots,N\}} Q_n y_n^i$$

- What is the sum in the optimised expression equal to?

Arrow securities

- Defining $a_n^i \equiv c_n^i - y_n^i$ the constraint can be rewritten as

$$c_0^i + \sum_{n \in \{1,\ldots,N\}} Q_n a_n^i = y_0^i$$

  - Since y_n^i are fixed, choosing c_n^i is equivalent to choosing a_n^i

- Can think of the agent as choosing c_0^i and holdings of assets a_n^i paying a unit of consumption only in state n (next period)
  - These assets are known as Arrow securities and their prices are denoted by Q_n. What is the unit of account?

Asset market completeness

- Asset market completeness implies no difference between
  - a dynamic stochastic model
  - a static model in which consumption at all possible future dates / states is chosen in the initial period
- But what does it imply about the number of Arrow securities?
  - Would you consider this to be a strong assumption?
- In summary, and as noted by Townsend (1979) (and many others), the insights of Arrow (1964) and Debreu (1959) are double-edged!
  - It seems there are few contingent dealings among agents relative to those suggested by the theory!
- We will stick to the complete markets assumption
  - Financial frictions consitute a popular deviation
  - Covered in more detail later in the course!

Solving the dynamic stochastic problem

- How can we quickly solve the two-period DSGE model?
- Optimal consumption levels are given by

$$c_0^i = \frac{1}{1+\beta}\left(y_0^i + \sum_{m\in\{1,\ldots,N\}} Q_m y_m^i\right)$$

$$c_n^i = \frac{\beta\pi_n/Q_n}{1+\beta}\left(y_0^i + \sum_{m\in\{1,\ldots,N\}} Q_m y_m^i\right)$$

  with Arrow security prices equal to

$$\forall n > 0 : Q_n = \beta\pi_n \frac{\mathbf{y}_0}{\mathbf{y}_n}$$

- To back out equilibrium security prices Q_n we need the aggregate endowments $\mathbf{y}_i$, discount factor β and state probabilities π_i

Solving the dynamic stochastic problem

- We have just solved a heterogenous agent DSGE model!
  - Can we say with certainty how much agent i will consume in the final period?
  - Can we say with certainty how much agent i will consume in state n in the final period?
- The solution is a conditional consumption plan for each agent i
  - Plan is time-consistent and expectations are rational!

Heterogeneous vs representative agent models

- We could also consider the following representative agent model

$$\max_{c_0, a_1, \ldots, a_n} \left\{ \log(\mathbf{c}_0) + \beta \sum_{n>0} \pi_n \log(\mathbf{c}_n) \right\}$$

$$s.t. \quad : \quad \mathbf{c}_0 + \sum_{n \in \{1,\ldots,N\}} Q_n \mathbf{a}_n = \mathbf{y}_0$$

  - Note that i has vanished, we have one agent only!

- What is the solution for $\mathbf{c}_i$? What is the implication for $\mathbf{a}_i$?
- The previous formulae for asset prices still apply, i.e.

$$\forall n > 0 : Q_n = \beta \pi_n \frac{\mathbf{y}_0}{\mathbf{y}_n}$$

- How are asset prices Q_n different in the representative agent model from the heterogenous agent one?

Notes on equivalence

- If all we're interested is aggregate prices then we can use the representative agent model...
  - We can think of there being a heterogenous agent economy in the background in which Arrow securities are actively traded
- Conditions under which the equivalence result holds were studied by Terence Gorman (Econometrica, 53)
  - Issue: the individual endowment distribution y_n^i should not matter for equilibrium prices
  - Idea: come up with conditions which guarantee that all agents, irrespective of wealth, chose the same bundle of goods
  - Necessary and sufficient condition: individual preferences admit Gorman-form indirect utility
- Assumption is satisfied by CRRA utility functions
  - log preferences are OK, but many other ones are not!

The consumption Euler equation

- We have previously shown that the price of the n'th Arrow security equals

$$\forall n \in \{1, \dots, N\} : Q_n = \beta \pi_n \frac{\mathbf{y}_0}{\mathbf{y}_n}$$

- How could you use this formula to determine the price of an asset which pays a unit of consumption with certainty in the final period?
  - Such an asset is known as a riskless real bond and its price equals Q

The consumption Euler equation

- Letting E_0 be the expectation operator, we have

$$Q = \sum_{m \in \{1,\ldots,N\}} Q_m = \sum_{m \in \{1,\ldots,N\}} \beta \pi_m \frac{\mathbf{y}_0}{\mathbf{y}_m} = \beta \mathbf{y}_0 E_0 \frac{1}{\mathbf{y}_{t=1}}$$

- These derivations were for log utility where $u'(c) = 1/c$; using market clearing ($\mathbf{y} = \mathbf{c}$) the general expression for Q is

$$Q = \beta \mathbf{y}_0 E_0 \frac{1}{\mathbf{y}_{t=1}} = \beta E_0 \frac{u'(\mathbf{c}_{t=1})}{u'(\mathbf{c}_0)}$$

  - This is the consumption Euler equation

The consumption Euler equation: intuition

- Define the net real interest rate r as

$$1 + r \equiv \frac{1}{Q}$$

- The consumption Euler equation can then be rewritten as

$$u'(\mathbf{c}_0) = \beta E_0 u'(\mathbf{c}_{t=1})(1+r)$$

  - The 'utility' cost of a marginal increase in saving: $u'(\mathbf{c}_0)$
  - The expected benefit: $\beta E_0 u'(\mathbf{c}_{t=1})(1+r)$

- What do higher real interest rates $r \uparrow$ imply for current ($\mathbf{c}_0$) and future ($\mathbf{c}_{t=1}$) consumption?
  - Higher real interest rates are thus contractionary

The Euler equation: link to monetary models

- In models with inflation, the Fisher parity (an identity linking real and nominal interest rates and inflation)

$$1 + r \equiv \frac{1 + i}{1 + \pi_{t=1}}$$

can be plugged into the consumption Euler equation, yielding

$$u'(\mathbf{c}_0) = \beta E_0 u'(\mathbf{c}_{t=1}) \frac{1 + i}{1 + \pi_{t=1}}$$

- By the exact same mechanism as previously, higher expected inflation ceteris paribus results in higher consumption today and lower future consumption!
- Importantly, increases in the nominal interest rate i would lead to lower consumption today, in line with the standard interest rate channel of monetary policy transmission
  - Caveat: expected inflation could respond to changes in i

DSGE models: Summary

- We started by solving a heterogenous-agent, static, deterministic, general equilibrium model
- We showed that when asset markets are complete, the setup can easily be made dynamic (i.e. account for many periods) and stochastic (i.e. account for uncertainty)
  - However, the assumption of complete markets seems counterfactual!
- We looked at what can be inferred about (expected) utility from axioms on choice / revealed preferences

DSGE models: Summary (ctd)

- We also showed that under Gorman-form utility functions our heterogenous agent model will display exactly the same asset price dynamics as a representative agent model
  - Using a representative agent model does not imply loss of generality $\Longrightarrow$ heterogeneity may not matter for some questions!
- Finally, we also derived the the Euler equation

$$u'(\mathbf{c}_0) = \beta E_0 u'(\mathbf{c}_{t=1})(1+r)$$

  - This suggests a link between marginal utility and the real interest rate
- We'll shortly analyse this simple DSGE model

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